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Variable exponent $p(x)$-Kirchhoff type problem with variable potential and convection

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Abstract: In this work, we study a nonlinear $p(x)$-Kirchhoff equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta_{p(x)}^K u + V(x)|u|^{p(x)-2}u = f(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

Using a topological approach based on Galerkin method, we get the existence of strong generalized solutions and weak solutions. The results obtained in the literature have been generalized.

Keywords: Variable exponent; Galerkin basis; Convection; Kirchhoff type term; Strong generalized solution; Weak solution

Mathematics Subject Classification. 35J60; 35J70; 58E05

1. Introduction and main results

As we all know, the mechanisms of fluid movement are very important in dealing with the problem of diffusion. In the study of porous media such as liquids and gases, there is a widely studied phenomenon called convection. In short, convection occurs when energy is transferred by moving particles. It mainly occurs when the temperature gradient exceeds a certain threshold. To study this phenomenon, we have introduced a reaction term $f(x,y,z)$, which depends on the gradient. On the other hand, we mainly consider the problem involving Kirchhoff type term. Similarly, the research on Kirchhoff type problems comes from physical applications, which is related to processes. It is known that Kirchhoff type problems is first studied by Kirchhoff [1] when he investigated an extension of the D'Alembert wave equation for free vibrations of elastic strings.

In this paper, we study the following kind of $p(x)$-Kirchhoff type problem with Dirichlet boundary condition, potential term and convection in the reaction term:

$$\begin{cases} -\Delta_{p(x)}^K u + V(x)|u|^{p(x)-2}u = f(x,u,\nabla u), & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a Carathéodory function, $V : \bar{\Omega} \to (0, +\infty)$ and $p : \bar{\Omega} \to (1, +\infty)$ is a Lipschitz function, and satisfies

$$1 < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x) < +\infty.$$

$\Delta_{p(x)}$ denotes the $p(x)$-Laplace differential operator defined as follows:

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

and the Kirchhoff type term is of the following form:

$$K(p, u) = a - b \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx, \quad a, b > 0. \tag{1.2}$$

Hence, the $p(x)$-Kirchhoff type operator denoted by $\Delta^K_{p(x)}$ is defined as follows:

$$\Delta^K_{p(x)} = K(p, u)\Delta_{p(x)} u = \left(a - b \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u), \quad u \in W_0^{1,p(x)}(\Omega),$$

where $W_0^{1,p(x)}(\Omega)$ will be defined in Section 2.

Recently, there have been many studies on the reaction term $f(x, y)$. For example, Fan and Zhang [2] obtained the existence of weak solutions for a kind of $p(x)$-Laplace equation; Hamadni, Harrabi and Mtiri [3] obtained the existence of nontrivial weak solutions of the nonlocal Kirchhoff equation with perturbation term $\lambda|u|^{p(x)-2}u$; Ge, Zhang and Hou [4] obtained the existence of the Nehari-type ground state solution for a kind of superlinear $p(x)$-Laplace equations with potential V. There are also some studies on the gradient dependent reaction term $f(x, y, z)$. Faria, Miyagaki and Motreanu [5] studied the existence of positive solutions for a class of quasilinear elliptic equations with Dirichlet boundary conditions; Gasiński and Júnior [6] studied the existence of weak solutions for a kind of quasilinear elliptic equations with double phase phenomenon and a reaction term depeding on the geadient. Especially, when $V(x) = 0$ in (1.1), Vetro [7] studied a special kind of (1.1) and obtained some results by using topological approach and the theory of operators of monotone type. As pointed out by Vetro, the work obtained in [7] is the first attempt to consider (1.2) with a convection reaction.

Since problem (1.1) has a reaction relying on the gradient and a variable potential, we cannot use the classic variational methods (such as mountain pass theorem) in the analysis of it. But we can borrow the idea of [7], that is using a topological method, which is based on fixed-point arguments and the theory of operators with monotype features.

We call that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1) if

$$\langle -\Delta^K_{p(x)} u, v\rangle + \int_\Omega V(x) \, |u|^{p(x)-2} \, uvdx = \int_\Omega f(x, u, \nabla u)vdx.$$

We know that if $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1), then there exists $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$ such that:

(1) $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$
(2) $-\Delta_p^K u_n + V(x)|u|^{p(x)-2}u - f(x,u_n,\nabla u_n) \rightharpoonup 0$ in $W^{-1,p'(x)}(\Omega)$, as $n \to +\infty$
(3) $\lim_{n\to+\infty}\langle -\Delta_p^K u_n, u_n - u\rangle = 0$.

Such a kind of solution ($u \in W_0^{1,p(x)}(\Omega)$ satisfying (1), (2), (3) above), is known as a strong generalized solution to problem (1.1), by the terminology of Motreanu [8]. Therefore, the set of weak solutions to (1.1) is a subset of the generalized solutions to (1.1)(it follows choosing $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$ with $u_n := u$ for all $n \in \mathbb{N}$).

Therefore, it becomes a natural question to derive the weak solution from the strong generalized solution. Vetro address this question in [7] with condition:

$$\int_\Omega |\nabla u_n|^{p(x)}dx \nrightarrow \frac{a}{b} \text{ as } n \to +\infty, \{u_n\} \subseteq W_0^{1,p(x)}(\Omega). \tag{1.3}$$

First of all, we make the assumptions about the exponent p.
(p) There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x : \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x + z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.
(p′) $p \in C(\overline{\Omega})$ is finite with $p^+ < 2p^-$.

Hypothesis (p) is significant, since according to [9], which leads to the Rayleigh quotient

$$\lambda = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx} > 0. \tag{1.4}$$

Throughout this paper, we assume that the nonlinear term $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathédory function. The following hypotheses are required in the superlinear case.
(f) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathédory function such that the following two case hold:
(i) there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $1 < \alpha(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{otherwise} \end{cases}$, and $c > 0$ such that

$$|f(x,y,z)| \le c(\sigma(x) + |y|^{\alpha(x)-1} + |z|^{\frac{p(x)}{\alpha'(x)}}), \text{ for a.e. } x \in \Omega, \text{all } y \in \mathbb{R} \text{ and } z \in \mathbb{R}^N;$$

(ii) there exist $a_0 \in L^1(\Omega)$ and $b_1, b_2 \ge 0$ such that

$$|f(x,y,z)y| \le a_0(x) + b_1|y|^{p(x)} + b_2|z|^{p(x)}, \text{ for a.e. } x \in \Omega, \text{all } y \in \mathbb{R} \text{ and } z \in \mathbb{R}^N.$$

We assume that the potential V satisfies the following hypotheses:
(V) $V \in C(\mathbb{R}^N)$ and $0 < V^- := \inf_{x\in\mathbb{R}^N} V(x) \le \sup_{x\in\mathbb{R}^N} V(x) := V^+ < +\infty$.

We are now in a position to state the main results of this paper.

Theorem 1.1. *If hypotheses* (V)*,* (p) *and* (f) *hold, then problem* (1.1) *admits a strong generalized solution* $u \in W_0^{1,p(x)}(\Omega)$.

Theorem 1.2. *If hypotheses* (V)*,* (p′) *and* (f) *hold, then problem* (1.1) *admits a strong generalized solution* $u \in W_0^{1,p(x)}(\Omega)$.

Theorem 1.3. *Let $u \in W_0^{1,p(x)}(\Omega)$ be a strong generalized solution to problem* (1.1)*, associated to the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ satisfying* (1.3)*. If hypotheses* (V) *and* (f) *hold, then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem* (1.1)*.*

Remark 1.4. *When $V(x) \equiv 0$, Vetro* [7] *first studied a kind of problem* (1.1) *with convection and obtained some existence results for two notions of solutions, by applying a topological method. Since problem* (1.1) *involves variable potential $V(x)$, it is somewhat difficult to deal with the existence of solutions for problem* (1.1)*. To conquer this difficulty, we need some conditions on $V(x)$, and using the method in* [7]*, we establish some results for problem* (1.1)*. Hence, the obtained results of this paper can be seem as some generalization of relative works in* [7]*.*

This paper is organized as follows. In Section 2, we introduce some preliminaries konwledge of variable exponent spaces and give some preliminary lemmas facts which are needed to prove our results. And in Section 3, we present the proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3.

2. Preliminaries

In order to discuss problem (1.1), we first recall some necessary facts on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}$ which are called variable exponent Sobolev space. Let

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}), p(x) > 1, \text{for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real} - \text{valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} := |u|_{p(x)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}$ is defined by

$$W^{1,p(x)} = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

whose norm is given by

$$\|u\|_{W^{1,p(x)}} = |u|_{p(x)} + |\nabla u|_{p(x)}. \tag{2.1}$$

Lemma 2.1.[7] *Let X, Y be two Banach space with $X \subseteq Y$. If X is dense in Y and the embedding is continuous, then the embedding $Y^* \subseteq X^*$ is also continuous. Additionally, the reflexive of X implies that Y^* is dense in X^*.*

Proposition 2.2. (Poincaré Inequality) [10] *There is a constant $C > 0$ such that*

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \ \forall \ u \in W_0^{1,p(x)}(\Omega), \tag{2.2}$$

where $W_0^{1,p(x)}(\Omega)$ is the $W^{1,p(x)}$-norm closure of $C_0^\infty(\Omega)$.

Remark 2.3. *By Proposition 2.2, we know that* $|\nabla u|_{p(x)}$ *and* $\|u\|_{W^{1,p(x)}(\Omega)}$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$*, so we can replace* $\|u\|_{W^{1,p(x)}(\Omega)}$ *by* $|\nabla u|_{p(x)}$.

Proposition 2.4. [11] *The functional* $\rho_p(u) : L^{p(x)}(\Omega) \to \mathbb{R}$ *defined by*

$$\rho_p(u) = \int_\Omega |u|^{p(x)} dx,$$

has the following properties:

(i). $|u|_{p(x)(\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho_p(u) < 1 (= 1, > 1)$;

(ii). $if\ |u|_{p(x)} > 1,\ then\ |u|^{p^-}_{p(x)} \leq \rho_p(u) \leq |u|^{p^+}_{p(x)}$;

(iii). $if\ |u|_{p(x} < 1,\ then\ |u|^{p^+}_{p(x)} \leq \rho_p(u) \leq |u|^{p^-}_{p(x)}$.

Remark 2.5. *The following inequality can be calculated from Proposition 2.4:*

$$|u|^{p^-}_{p(x)} - 1 \leq \rho_p(u) \leq |u|^{p^+}_{p(x)} + 1. \tag{2.3}$$

Moreover, from (2.3)*, we have the following results by some easy calculations,*

$$||u|^{p(x)-1}|^{(p')^-}_{p'(x)} \leq \int_\Omega (|u|^{p(x)-1})^{\frac{p(x)}{p(x)-1}} dx + 1 \leq \left[|u|^{p^+}_{p(x)} + 1\right] + 1,$$

where we use the fact that $u \in L^{p(x)}(\Omega)$ *implies that* $|u|^{p(x)-1} \in L^{p'(x)}(\Omega)$*. Thus, we can obtain that*

$$\left||u|^{p(x)-1}\right|_{p'(x)} \leq 2 + |u|^{p^+}_{p(x)}. \tag{2.4}$$

Following a similar argument, we can obtain the following inequality:

$$\left||\nabla u|^{\frac{p(x)}{\alpha'(x)}}\right|_{\alpha'(x)} \leq 2 + ||\nabla u||^{p^+}_{p(x)},\ \alpha \in C(\overline{\Omega}) \text{ with } \alpha(x) > 1 \text{ for all } x \in \overline{\Omega}. \tag{2.5}$$

Proposition 2.6. [7] *Let* $(X, \|\cdot\|)_X$ *be a normed finite-dimensional space and let* $T : X \to X^*$ *be a continuous map. If there exists some* $R > 0$ *such that*

$$\langle T(v), v\rangle \geq 0, \quad \forall\ v \in X \text{ with } \|v\|_X = R,$$

then the equation $T(v) = 0$ *has a solution* $u \in X$ *such that* $R \geq \|u\|_X$.

Proposition 2.7. [11] *The conjugte space of* $L^{p(x)}(\Omega)$ *is* $L^{p'(x)}(\Omega)$*, where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$*. For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$*, we have*

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}.$$

Next, we give some lemmas needed to prove our results.

Lemma 2.8. *If hypothesis* (f)(i) *holds, then for all* $u, v \in W_0^{1,p(x)}(\Omega)$*, there is some* $c > 0$ *such that the following inequality holds:*

$$\left|\int_\Omega f(x, u, \nabla u) v dx\right| \leq 2c|v|_{\alpha(x)} \left[|\sigma|_{\alpha'(x)} + |u|^{\alpha^+}_{\alpha(x)} + |\nabla u|^{p^+}_{p(x)} + 4\right].$$

Proof. For all $u, v \in W_0^{1,p(x)}(\Omega)$ and some $c > 0$, by (f)(i), Hölder inequality, (2.4) and (2.5), we can obtain

$$\begin{aligned}\left|\int_\Omega f(x,u,\nabla u)v dx\right| &\leq c\int_\Omega \left[|\sigma(x)| + |u|^{\alpha(x)-1} + |\nabla u|^{\frac{p(x)}{\alpha'(x)}}\right]|v|dx \\ &\leq 2c|v|_{\alpha(x)}\left[|\sigma|_{\alpha'(x)} + \left||u|^{\alpha(x)-1}\right|_{\alpha'(x)} + \left||\nabla u|^{\frac{p(x)}{\alpha'(x)}}\right|_{\alpha'(x)}\right]. \qquad (2.6)\end{aligned}$$

The proof is completed. □

Now, we define $N_f^* : W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ to be the Nemitsky map corresponding to the Carathéodory function f, that is

$$N_f^*(u) = f(x, u, \nabla u), \quad \forall\ u \in W_0^{1,p(x)}(\Omega).$$

By (f)(i) (see[12]), we know that N_f^* is bounded and continuous. Afterwards, we consider the operator $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$N_f = i^* \circ N_f^*,$$

where $i^* : L^{\alpha'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a continuous embedding (see Lemma 2.1). It follows that N_f is bounded and continuous.

On the other hand, by Lemma 2.8, for all $v \in W_0^{1,p(x)}(\Omega)$ and some $c > 0$, we have

$$\|N_f(v)\|_{W^{-1,p'(x)}(\Omega)} \leq 2c\left[|\sigma|_{\alpha'(x)} + |v|_{\alpha(x)}^{\alpha^+} + |\nabla v|_{p(x)}^{p^+} + 4\right].$$

Since $W_0^{1,p(x)}(\Omega)$ is a separable Banach space, we can find a Galerkin basis $\{X_n\} \subset W_0^{1,p(x)}(\Omega)$ such that

(i) $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$;

(ii) $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$;

(iii) $\overline{\bigcup_{n=1}^\infty X_n} = W_0^{1,p(x)}(\Omega)$.

Lemma 2.9. *Let $\{X_n\}$ be a Galerkin basis of $W_0^{1,p(x)}(\Omega)$. If hypotheses* (p), (V) *and* (f) *hold, then for any $n \in \mathbb{N}$, we can find $u_n \in X_n$ such that*

$$\langle -\Delta_p^K u_n, v\rangle + \int_\Omega V(x)|u_n|^{p(x)-2}u_n v dx = \int_\Omega f(x, u_n, \nabla u_n)v dx, \quad \forall\ v \in X_n. \qquad (2.7)$$

Proof. Fixed $n \in \mathbb{N}$, let $T_n : X_n \to X_n^*$ be the operator defined by

$$\langle T_n(u), v\rangle = \langle -\Delta_p^K u, v\rangle + \int_\Omega V(x)|u|^{p(x)-2}uv dx - \int_\Omega f(x,u,\nabla u)v dx, \quad \forall\ v \in X_n.$$

By (1.4), hypotheses (f)(ii) and (V), we can get

$$\langle -T_n(v), v\rangle \quad = \quad \left(b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx - a\right)\int_\Omega |\nabla v|^{p(x)}dx$$

$$
\begin{aligned}
& -\int_\Omega V(x)|v|^{p(x)}dx + \int_\Omega f(x,v,\nabla v)vdx \\
\geq\ & \left(b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx - a\right)\int_\Omega |\nabla v|^{p(x)}dx \\
& -\int_\Omega V(x)|v|^{p(x)}dx - \int_\Omega |f(x,v,\nabla v)v|dx \\
\geq\ & \frac{b}{p^+}\left(\int_\Omega |\nabla v|^{p(x)}dx\right)^2 - a\int_\Omega |\nabla v|^{p(x)}dx - \lambda^{-1}V^+\int_\Omega |\nabla v|^{p(x)}dx \\
& -\int_\Omega |a_0(x)|dx - b_1\int_\Omega |v|^{p(x)}dx - b_2\int_\Omega |\nabla v|^{p(x)}dx \\
\geq\ & \frac{b}{p^+}\rho_p^2(\nabla v) - a\rho_p(\nabla v) - \lambda^{-1}V^+\rho_p(\nabla v) \\
& -\|a_0\|_{L^1(\Omega)} - b_1\lambda^{-1}\rho_p(\nabla v) - b_2\rho_p(\nabla v).
\end{aligned} \tag{2.8}
$$

If $\rho_p(\nabla v) > 1$, then from (2.8), we have

$$
\begin{aligned}
\langle -T_n(v), v\rangle &\geq \frac{b}{p^+}\rho_p^2(\nabla v) - \left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}\right)\rho_p(\nabla v) \\
&= \left[\frac{b}{p^+}\rho_p(\nabla v) - \left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}\right)\right]\rho_p(\nabla v), \\
\Rightarrow\ \ \langle -T_n(v), v\rangle \geq 0\ \ if\ \rho_p(\nabla v) &\geq \frac{p^+}{b}\left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}\right).
\end{aligned}
$$

Fixed $R > \max\left\{\left[\frac{p^+}{b}\left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}\right)\right]^{1/p^-}, 1\right\}$, then for any $v \in X_n$ with $\|v\| = R$, we have

$$
\langle -T_n(v), v\rangle \geq 0.
$$

Combining Proposition 2.6, the equation $T_n(u) = 0$ has a solution $u_n \in X_n$. The proof is completed. □

Remark 2.10. *From (2.3), we can know that any* $S \subseteq W_0^{1,p(x)}(\Omega)$ *is bounded if there exists a constant* $C > 0$ *such that* $\rho_p(\nabla u) \leq C$ *for all* $u \in S$.

Let $\{u_n\} \subset \cup_{n=1}^\infty X_n$ be the sequence mentioned in the proof of Lemma 2.9. Then, we have the following lemma which is used to prove that the boundedness of such a sequence in $W_0^{1,p(x)}(\Omega)$.

Lemma 2.11. *If hypotheses* (p), (f) *and* (V) *hold, then* $\{u_n\} \subset \cup_{n=1}^\infty X_n$ *is bounded in* $W_0^{1,p(x)}(\Omega)$.

Proof. If $\rho_p(\nabla u_n) \leq 1$ for all $n \in \mathbb{N}$, by Remark 2.10, we know that the sequence $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore, we assume that there exists some $n \in \mathbb{N}$ such that $\rho_p(\nabla u_n) > 1$. Replacing v in (2.7) with u_n, combining (1.4) and hypothesi (f)(ii), we get

$$
\begin{aligned}
\frac{b}{p^+}\rho_p^2(\nabla u_n) &= \frac{b}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 \\
&\leq a\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega V(x)|u_n|^{p(x)}dx - \int_\Omega f(x,u_n,\nabla u_n)u_n dx \\
&\leq a\rho_p(\nabla u_n) + \lambda^{-1}V^+\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega |f(x,u_n,\nabla u_n)u_n|dx
\end{aligned}
$$

$$\begin{aligned}
&\leq a\rho_p(\nabla u_n) + \lambda^{-1}V^+\rho_p(\nabla u_n) + \int_\Omega \left[|a_0(x)| + b_1|u_n|^{p(x)} + b_2|\nabla u_n|^{p(x)}\right] dx \\
&\leq \left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2\right)\rho_p(\nabla u_n) + \|a_0\|_{L^1(\Omega)}.
\end{aligned}$$

From the above, there is

$$\rho_p(\nabla u_n) \leq \frac{p^+}{b}(a + \lambda^{-1}V^+ = b_1\lambda^{-1} + b_2 + \|a_0\|_{L_1(\Omega)}). \tag{2.9}$$

Thus, from (2.9) and $\|\nabla u_n\|_{L^{p(x)}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$, we have

$$\rho_p(\nabla u_n) \leq \max\left\{\frac{p^+}{b}\left(a + \lambda^{-1}V^+ + b_1\lambda^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}\right), 1\right\},$$

which implies that $\{u_n\} \subset \cup_{n=1}^\infty X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. The proof is completed. $\square$

Lemma 2.12. *Let $\{X_n\}$ be a Galerkin basis of $W_0^{1,p(x)}(\Omega)$. If hypotheses* (p′), (V) *and* (f) *hold, then for any $n \in \mathbb{N}$, we can find $u_n \in X_n$ such that*

$$\langle -\Delta_p^K u_n, v\rangle + \int_\Omega V(x)|u_n|^{p(x)-2}u_n v dx = \int_\Omega f(x, u_n, \nabla u_n) v dx, \quad \forall\, v \in X_n. \tag{2.10}$$

Proof. Similar to the proof of Lemma 2.9, fixed $n \in \mathbb{N}$, we consider that the operator $T_n : X_n \to X_n^*$ defined by

$$\langle T_n(u), v\rangle = \langle -\Delta_p^K u, v\rangle + \int_\Omega V(x)|u|^{p(x)-2}uvdx - \int_\Omega f(x, u, \nabla u)vdx, \quad \forall\, v \in X_n.$$

By (1.3), hypotheses(f)(ii) and (V), we have

$$\begin{aligned}
\langle -T_n(v), v\rangle \geq &\frac{b}{p^+}\left(\int_\Omega |\nabla v|^{p(x)}dx\right)^2 - (a + b_2)\int_\Omega |\nabla v|^{p(x)}dx \\
&- (b_1 + V^+)\int_\Omega |v|^{p(x)}dx - \|a_0\|_{L^1(\Omega)}, \quad \forall\, v \in X_n.
\end{aligned}$$

If $\|v\| = \|\nabla v\|_{L^{p(x)}(\Omega)} > 1$, we get

$$\begin{aligned}
\langle -T_n(v), v\rangle \geq &\frac{b}{p^+}\|v\|^{2p^-} - (a + b_2)\|v\|^{p^+} - \|a_0\|_{L^1(\Omega)} \\
&- (b_1 + V^+)\max\left\{|v|_{p(x)}^{p^+}, |v|_{p(x)}^{p^-}\right\} \\
\geq &\frac{b}{p^+}\|v\|^{2p^-} - \left[a + b_2 + (b_1 + V^+)C_2 + \|a_0\|_{L^1(\Omega)}\right]\|v\|^{p^+} \\
= &\left\{\frac{b}{p^+}\|v\|^{2p^- - p^+} - [a + b_2 + (b_1 + V^+)C_2 + \|a_0\|_{L^1(\Omega)}]\right\}\|v\|^{p^+},
\end{aligned}$$

where $C_2 = C_2(p^-, p^+, c_1) > 0$.

Fixed $R > \max\left\{\left[\frac{b}{p^+}\|v\|^{2p^- - p^+} - (a + b_2 + b_1C_2 + V^+C_2 + \|a_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}, 1\right\}$, then for any $v \in X_n$ with $\|v\| = R$, we have

$$\langle -T_n(v), v\rangle \geq 0.$$

Combining Proposition 2.6, we can deduce that the equation $T_n(u) = 0$ has a solution $u_n \in X_n$. And the proof is completed. □

Lemma 2.13. *If hypotheses* (p′), (f) *and* (V) *hold, then* $\{u_n\} \subset \cup_{n=1}^{\infty} X_n$ *is bounded in* $W_0^{1,p(x)}(\Omega)$.

Proof. If $\|\nabla u_n\|_{L^{p(x)}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$, then the sequence $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore, we assume that there exists some $n \in \mathbb{N}$ such that $\|\nabla u_n\|_{L^{p(x)}(\Omega)} > 1$. Replacing v in (2.7) with u_n, combining (1.4) and hypothesi (f)(ii), we get

$$\begin{aligned} \frac{b}{p^+}\|\nabla u_n\|_{L^{p(x)}(\Omega)}^{2p^-} &\leq \frac{b}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 \\ &\leq a\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega V(x)|u_n|^{p(x)}dx - \int_\Omega f(x,u_n,\nabla u_n)u_n dx \\ &\leq \left[a + b_2 + \left(b_1 + V^+\right) C_2\right] |\nabla u_n|_{p(x)}^{p^+} + \|a_0\|_{L^1(\Omega)}. \end{aligned} \tag{2.11}$$

From (2.11), we have

$$|\nabla u_n|_{p(x)} \leq [a + b_2 + (b_1 + V^+)C_2 + \|a_0\|_{L^1(\Omega)}]^{1/(2p^- - p^+)}. \tag{2.12}$$

Thus, from (2.12) and $\|\nabla u_n\|_{L^{p(x)}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$, we have

$$|\nabla u_n|_{p(x)} \leq \max\left\{\left[a + b_2 + \left(b_1 + V^+\right) C_2 + \|a_0\|_{L^1(\Omega)}\right]^{1/(2p^- - p^+)}, 1\right\}.$$

Hence, $\{u_n\} \subset \cup_{n=1}^{\infty} X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. The proof is completed. □

3. Proofs of Main Results

Proof of Theorem1.1. From Lemma 2.13, we see that $\{u_n\} \subset \cup_{n=1}^{\infty} X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. Since $W_0^{1,p(x)}(\Omega)$ is a reflexive space, we can assume that there exists $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \to u \text{ in } L^{\alpha(x)}(\Omega) \text{ and } u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega). \tag{3.1}$$

On the other hand, we know that the Nemitsky map is bounded, so we have

$$\{N_f(u_n)\} \text{ is bounded in } W^{-1,p'(x)}(\Omega).$$

Next, we consider $N_V^* : W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ which is defined as:

$$N_V^* = V(x)|u|^{p(x)-2}u.$$

By hypothesis (V), (1.4) and Hölder inequality, we have

$$\begin{aligned} \int_\Omega V(x)|u|^{p(x)-2}u &\leq V^+\int_\Omega |u|^{p(x)-1}dx \\ &\leq V^+\left(\int_\Omega |u|^{p(x)}dx\right)^{\frac{p(x)-1}{p(x)}}\left(\int_\Omega 1^{\frac{p(x)}{p(x)-1}}\right)^{\frac{p(x)-1}{p(x)}} \\ &\leq V^+[\lambda^{-1}\rho_p(\nabla u)]^{\frac{p(x)-1}{p(x)}}[\mu(\Omega)]^{\frac{p(x)-1}{p(x)}}, \end{aligned}$$

where $\mu(\Omega)$ is the Lebesgue measure on $\mathbb{R}^N$. Since $u \in W_0^{1,p(x)}(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^N$, N_V^* is bounded and continuous. Now, we consider the operator $N_V : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$N_V = i^* \circ N_V^*.$$

It follows from Lemma 2.1 that N_V is bounded and continuous, that is

$$\{V(x)|u_n|^{p(x)-2}u_n\} \text{ is bounded in } W^{-1,p'(x)}(\Omega).$$

The boundedness of the operator $-\Delta^K_{p(x)}$ implies that

$$\{-\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n)\} \text{ is bounded in } W^{-1,p'(x)}(\Omega).$$

By the reflexivity of the space $W^{-1,p(x)}(\Omega)$, for some $\tau \in W^{-1,p(x)}(\Omega)$, we have

$$-\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n) \rightharpoonup \tau \text{ in } W^{-1,p(x)}(\Omega), \tag{3.2}$$

which is true at least for a relabeled subsequence of $\{-\Delta^K_{p(x)} + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n)\}$.

Choosing $v \in \cup_{n=1}^{\infty} X_n$, then we can find $n(v) \in \mathbb{N}$ such that $v \in X_{n(v)}$. By Lemma 2.9, we can know that (2.7) holds for all $n \geq n(v)$. Taking the limit as $n \to +\infty$ in (2.7), we can obtain

$$\langle \tau, v\rangle = 0, \ \forall\ v \in \cup_{n=1}^{\infty} X_n.$$

Since $\{X_n\}$ is a Galerkin basis, $\cup_{n=1}^{\infty} X_n$ is dense in $W_0^{1,p(x)}(\Omega)$, then we deduce that $\tau = 0$. Hence, from (3.2), we have

$$-\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n) \rightharpoonup 0 \text{ in } W^{-1,p(x)}(\Omega). \tag{3.3}$$

Next, we replace v with u_n in (2.7) and get

$$\begin{aligned} a\int_\Omega |\nabla u_n|^{p(x)}dx &= b\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 - \int_\Omega V(x)|u_n|^{p(x)}dx \\ &\quad + \int_\Omega f(x,u_n,\nabla u_n)dx, \quad \forall\ n \in \mathbb{N}. \end{aligned} \tag{3.4}$$

From (3.3), we have

$$\langle -\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n), u\rangle \to 0 \text{ as } n \to +\infty. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\langle -\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n), u_n - u\rangle \to 0 \text{ as } n \to +\infty. \tag{3.6}$$

From Lemma 2.8, replacing u with u_n and let $v = (u_n - u)$, for some $c > 0$ and all $n \in \mathbb{N}$, we have

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n - u)dx\right|$$

$$\leq \quad 2c|u_n - u|_{\alpha(x)} \left[|\sigma|_{\alpha'(x)} + |u_n|_{\alpha(x)}^{\alpha^+} + |\nabla u_n|_{p(x)}^{p^+} + 4\right]. \tag{3.7}$$

By Lemma 2.11, $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, and therefore $\{u_n\}$ is bounded in $L^{\alpha(x)}(\Omega)$. Similarly, $\{|\nabla u_n|\}$ is also bounded in $L^{p(x)}(\Omega)$. Thus, from (3.7), we obtain

$$\left|\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx\right| \leq C_1|u_n - u|_{\alpha(x)}, \tag{3.8}$$

where $C_1 > 0$. By $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ in (3.5), it follows from (3.8) that

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx = 0. \tag{3.9}$$

From (3.5), (3.6) and (3.9), we have

$$\lim_{n\to+\infty} \langle -\Delta^K_{p(x)} u_n + V(x)|u_n|^{p(x)-2}u_n, u_n - u\rangle = 0. \tag{3.10}$$

Therefore, from (3.1), (3.5) and (3.10), we can know that $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to problem (1.1). □

Proof of Theorem1.2. The proof here only requires the replacement of Lemma 2.9 and Lemma 2.11 in the proof of Theorem 1.1 with Lemma 2.12 and Lemma 2.13, respectively. So the proof here is omitted.

Proof of Theorem1.3. By Hölder inequality and (3.7), we have

$$\begin{aligned}\left|\int_\Omega V(x)|u_n|^{p(x)-2}u_n(u_n - u)dx\right| &\leq V^+ \int_\Omega |u_n|^{p(x)-1}|u_n - u|dx \\ &\leq V^+ \left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}} |u_n - u|_{p(x)} \\ &\to 0 \text{ as } n \to +\infty,\end{aligned}$$

and thus,

$$\lim_{n\to+\infty} \int_\Omega V(x)|u_n|^{p(x)-2}u_n(u_n - u)dx = 0. \tag{3.11}$$

In addition, since the sequence $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$ is bounded, there exists at least a relabeled subsequence, which we assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \neq \frac{a}{b}, \quad \forall\, n \in \mathbb{N},$$

and

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to t_0 \neq \frac{a}{b}, \quad \text{as } n \to +\infty, \text{ for some } t_0 > 0. \tag{3.12}$$

From (3.12), we have

$$a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to a - bt_0 \neq 0.$$

Hence, we can find $\delta > 0$ such that

$$\left|a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right| \geq \delta > 0, \quad \forall\, n \in \mathbb{N}. \tag{3.13}$$

Since the sequence $\{a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\}$ is bounded, combining (3.10), (3.11) and (3.12), we get

$$\lim_{n\to+\infty}\langle -\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n, u_n - u\rangle = 0. \tag{3.14}$$

By (3.14), we have

$$\lim_{n\to+\infty}\left[\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\langle -\Delta_{p(x)}u_n, u_n - u\rangle + \int_\Omega V(x)|u_n|^{p(x)-2}u_n(u_n - u)dx.\right] = 0. \tag{3.15}$$

Hence, from (3.11) and (3.15), we obtain

$$\lim_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle = 0. \tag{3.16}$$

Since $-\Delta_{p(x)}$ has the $(S)_+$-property, from (3.16), we have

$$u_n \to u \ \text{ in } W_0^{1,p(x)}(\Omega).$$

By the definition of strong generalized solution, we have

$$-\Delta^K_{p(x)}u_n + V(x)|u_n|^{p(x)-2}u_n - N_f(u_n) \rightharpoonup 0 \text{ in } W^{-1,p(x)}(\Omega),$$

and

$$-\Delta^K_{p(x)}u + V(x)|u|^{p(x)-2}u - N_f(u) = 0,$$

which implies that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1). $\square$

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